Functional Analysis HW 7

Deadline: 13 Apr 2017

1. Let X be a complex inner product space. Let $(y_1, y_2, ...)$ be a sequence of non-zero vectors in X. Set $u_1 := y_1/\|y_1$ and u_{n+1} is inductively defined by

$$u_{n+1} = \frac{y_{n+1} - \sum_{k=1}^{n}(y_{n+1}, u_k)u_k}{\|y_{n+1} - \sum_{k=1}^{n}(y_{n+1}, u_k)u_k\|}$$

Show that (u_n) forms an orthonormal sequence in X.
(The above construction is called the *Gram-Schmidt process*).

2. A normed space X is said to be locally uniformly convex if x and x_n in $S_X := \{y \in X : \|y\| = 1\}$, for $n = 1, 2...$ which satisfy $\|\frac{x_n+x}{2}\| \to 1$, will imply that $\|x_n - x\| \to 0$. (Try to understand the geometric meaning by drawing a picture on paper).

   (i) Is ℓ^∞ a locally uniformly convex?

   (ii) Show that every inner product space is locally uniformly convex.

   Remark
   In fact, the Banach space ℓ^p is locally uniformly convex for $1 < p < \infty$. It can be shown by the well known *Clarkson's inequality*:

$$\|\frac{x+y}{2}\|_p^p + \|\frac{x-y}{2}\|_p^p \leq \frac{1}{2}\|x\|_p^p + \frac{1}{2}\|y\|_p^p$$

   for all $x, y \in \ell^p$.